KU-TP 004
hep-th/0607105

On the No-ghost Theorem in String Theory

Kazuyuki Furuuchi[a,1] and Nobuyoshi Ohta[b,2]

[a] *Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India*

[b] *Department of Physics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*

Abstract

We give a simple proof of the no-ghost theorem in the critical bosonic string theory by using a similarity transformation.

[1] e-mail address: furuuchi@mri.ernet.in
[2] e-mail address: ohtan@phys.kindai.ac.jp

Ten years ago, we proposed the idea of using a similarity transformation to give a simple proof of the no-ghost theorem in the critical bosonic string theory [1]. Recently there has appeared a proof of this theorem, which is slightly different from ours but uses also a similarity transformation [2]. In this short note, we present our proof with some corrections.

The BRST operator of the bosonic string is decomposed in the ghost zero modes as

$$Q_B = c_0 L_0 - b_0 M + d, \tag{1}$$

where

$$\begin{aligned} L_0 &= \alpha' p^2 + \sum_{n\neq 0}\left(\frac{1}{2} : \alpha^\mu_{-n}\alpha_{n\mu} : + n : b_{-n}c_n :\right) - 1, \quad M = \sum_{n\neq 0} n : c_{-n}c_n :, \\ d &= \sum_{n\neq 0} c_{-n}L_n - \frac{1}{2}\sum_{\substack{m,n\neq 0\\ m+n\neq 0}} (m-n) : c_{-m}c_{-n}b_{n+m} : . \end{aligned} \tag{2}$$

Here b_n and c_n are the ghost modes with $\{b_n, c_m\} = \delta_{n+m,0}$ and L_n is the Virasoro operator for the coordinates:

$$L_n = \frac{1}{2}\sum_m : \alpha^\mu_m \alpha_{n-m,\mu} : , \quad [\alpha^\mu_n, \alpha^\nu_m] = n\eta^{\mu\nu}\delta_{n+m,0}. \tag{3}$$

The nilpotency of Q_B is equivalent to the relations

$$d^2 = ML_0, \quad [d, L_0] = [d, M] = [M, L_0] = 0. \tag{4}$$

The physical state is defined by

$$Q_B|\text{phys}\rangle = 0, \tag{5}$$

namely as the cohomology of the nilpotent BRST operator Q_B. The no-ghost theorem claims that the space satisfying this condition does not involve states of negative norm.

Since $L_0 = \{b_0, Q_B\}$, the physical states obeying the condition (5) also satisfy

$$L_0|\psi\rangle = Q_B b_0|\psi\rangle. \tag{6}$$

Consequently any physical state is BRST-exact unless it satisfies the on-shell condition $L_0 = 0$. It is convenient to reduce the zero eigenspace of L_0 by restricting to the states annihilated by b_0. In this space, the physical state condition (5) reduces to

$$L_0|\text{phys}\rangle = 0, \quad b_0|\text{phys}\rangle = 0, \quad d|\text{phys}\rangle = 0. \tag{7}$$

Note that we have $d^2 = 0$ in this space due to the relation (4).

Now we choose our coordinate system such that $p^i = 0$ $(i = 1, \cdots, 24)$ and $p^+ \equiv \sqrt{\alpha'}(p^{25} + p^0) \neq 0$, and define

$$\alpha_n^{\pm} = \frac{1}{\sqrt{2}}(\alpha_n^{25} \pm \alpha_n^0), \tag{8}$$

which satisfy $[\alpha_n^{\pm}, \alpha_m^{\mp}] = n\delta_{n+m,0}$. We also introduce the degree for the oscillators as

$$\begin{aligned} \deg(\alpha_n^+, c_n) &= +1, \\ \deg(\alpha_n^-, b_n) &= -1, \end{aligned} \tag{9}$$

and define the degrees for other oscillators and the vacuum to be zero. The cohomology operator d is decomposed into components with definite degrees:

$$d = d_0 + d_1 + d_2, \tag{10}$$

where

$$\begin{aligned} d_0 &= p^+ \sum_{n\neq 0} c_{-n}\alpha_n^-, \\ d_1 &= \sum_{n,m,n+m\neq 0} c_{-n}\left[\alpha_{-m}^+\alpha_{n+m}^- + \frac{1}{2}\alpha_{-m}^i\alpha_{n+m}^i + \frac{1}{2}(m-n)c_{-m}b_{n+m}\right], \\ d_2 &= p^- \sum_{n\neq 0} c_{-n}\alpha_n^+. \end{aligned} \tag{11}$$

We note that the normal ordering is imposed in the original definition of the charges but it is not necessary here because all the mode operators (anti)commute due to the constraints on the sum. The nilpotency of the operator d gives

$$d_0^2 = \{d_0, d_1\} = \{d_0, d_2\} + d_1^2 = \{d_1, d_2\} = d_2^2 = 0. \tag{12}$$

The complication in the no-ghost theorem in string theory comes from the fact that, in addition to d_0 and d_2, which are second order in the mode operators, there are third-order terms in d_1. It was shown that the cohomology of d_0 can be extended to that of d by adding terms of higher degrees [3]. This procedure was given perturbatively, and it implies that there is a one-to-one correspondence between the cohomology of d_0 and that of d. However, this perturbative proof is somewhat indirect and cumbersome. The results naturally suggest that there is a similarity between the cohomologies of these operators.

Here we show that this is indeed the case by explicitly giving the similarity transformation which maps d into d_0,

$$e^R d e^{-R} = d_0, \tag{13}$$

up to terms trivial in $|\text{phys}\rangle$. This transformation is useful in the formulation of "universal string theory"[4], and we expect that it will be useful for other purposes.

After some investigation, we find that

$$\begin{aligned} R &= \frac{1}{p^+} \sum_{m,n,m+n\neq 0} \Big[\frac{m+n}{2nm} \alpha^+_{-m} \alpha^+_{-n} \alpha^-_{m+n} \\ &\quad + \frac{1}{2m} \alpha^+_{-m} \alpha^i_{-n} \alpha^i_{m+n} - \frac{n}{m} \alpha^+_{-m} b_{m+n} c_{-n} \Big], \end{aligned} \tag{14}$$

has the necessary properties [1]. It is then easy to show that

$$[R, d_0] = -d_1, \quad [R, d_1] = 2 d_2 \frac{L_0 - p^+ p^-}{p^+ p^-}, \quad [R, d_2] = 0. \tag{15}$$

In deriving these results, it must be noted that there are various terms which drop due to the symmetry of the coefficients, and special attention must be paid to determining which combinations of the suffices remain in the sum according to the restriction imposed. It appears that the second relation is singular for $p^- = 0$, but this is because it is written in terms of the operator d_2, which contains p^-, and it is actually a well-defined operator. The following relations should also be understood in this way.

The result (15) means that

$$e^R d e^{-R} = d + [R, d] + \frac{1}{2}[R, [R, d]] + \cdots = d_0 + d_2 \frac{L_0}{p^+ p^-}, \tag{16}$$

which reduces to Eq. (13) upon using the on-shell condition in Eq. (7). As a consistency check, we note that R commutes with L_0 and M, so $e^R d^2 e^{-R} = M L_0$ because of Eq. (4). It is easy to see that this is true for Eq. (16) due to Eq. (12) and $\{d_0, d_2\} = p^+ p^- M$.

If we define $K = \frac{1}{p^+} \sum_{n\neq 0} \alpha^+_{-n} b_n$, then $N_0 = \{d_0, K\}$ is the level operator for $\alpha^\pm_{-n}, b_{-n}$ and c_{-n}. The states $|\psi\rangle$ in the cohomology of d_0 satisfy

$$N_0 |\psi\rangle = d_0 K |\psi\rangle, \tag{17}$$

so that all the states are cohomologically trivial, unless they satisfy $N_0 = 0$, or they do not contain these modes. Thus the cohomology of d_0 is spanned by the transverse oscillators α^i_{-n} with positive norm, which is denoted by $|\mathcal{P}\rangle$.

According to Eq. (16), the physical states of the theory are then given as

$$|\mathrm{phys}\rangle = e^{-R}|\mathcal{P}\rangle. \tag{18}$$

When we expand the exponent on the right-hand side of this relation, terms of higher degree appear. Under conjugation, the degree does not change, and the inner product is nonvanishing only for the case in which the total degree is 0. This means that the only terms contributing to the inner product are those of degree 0 made of the transverse oscillators, and hence they give a positive norm space. This completes our simplified proof of the no-ghost theorem.

We expect that there would be no essential difficulty in extending our method to superstrings [5].

Acknowledgments

We would like to thank Y. Kazama and Y. Aisaka for pointing out a problem in the original proof in Ref. [1] and for valuable discussions. The work of N.O. was supported in part by a Grant-in-Aid for Scientific Research No. 16540250.

References

[1] N. Ohta and K. Furuuchi, Soryushiron Kenkyu (in Japanese) **94** (1996) 69 and to be published.

[2] Y. Aisaka and Y. Kazama, JHEP **0404** (2004) 070, hep-th/0404141.

[3] M. Kato and K. Ogawa, Nucl. Phys. **B212** (1983) 443.

[4] H. Ishikawa and M. Kato, Mod. Phys. Lett. **A9** (1994) 725, hep-th/9311139; N. Ohta and J. L. Petersen, Phys. Lett. **B325** (1994) 67, hep-th/9312187; F. Bastianelli, N. Ohta and J. L. Petersen, Phys. Lett. **B327** (1994) 35, hep-th/9402042; Phys. Rev. Lett. **73** (1994) 1199, hep-th/9403150; F. Bastianelli and N. Ohta, Phys. Lett. **B348** (1995) 411, hep-th/9411156; H. Kunitomo, M. Sakaguchi and A. Tokura, Int. Jour. Mod. Phys. **A10** (1994) 827, hep-th/9403086; N. Ohta and T. Shimizu, Phys. Lett. **B355** (1995) 127, hep-th/9504099; N. Ohta, Prog. Theor. Phys. Suppl. **123** (1996) 411.

[5] N. Ohta, Phys. Rev. **D33** (1986) 1681; Phys. Lett. **B179** (1986) 347;
M. Ito, T. Morozumi, S. Nojiri and S. Uehara, Prog. Theor. Phys. **75** (1986) 934.

[6] W. Siegel, hep-th/9912205.

Note added: After this paper was submitted to the arXiv, we were informed that a discussion of a similarity transformation based on the representations of Poincaré group is given in Ref. [6]. We thank W. Siegel for pointing this out.